A New Generalization of the Exponentiated Gumbel Type-2 distribution with Application to Reliability data

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DOI: 10.56201/ijasmt.vol.11.no3.2025.pg1.13

Abstract
The addition of an extra parameter to standard distributions is a common technique in statistical theory. This study introduces a new generalization of the Exponentiated Gumbel distribution named alpha power exponentiated Gumbel Type-2 ($APEGT-2$) distribution. The $APEGT-2$ allows for a significant amount of versatility in modeling various data forms as it accommodates upside-down bathtubs, decreasing, and reversed-J shapes for hazard rate function. Some of the $APEGT-2$'s mathematical properties are derived in close forms. The maximum likelihood estimation technique was used for the purpose of estimation. An application to epoxy data demonstrate the flexibility of the $APEGT-2$ model compared to other models in the study.

Keywords: *Alpha Power Exponentiated Gumbel Type-2 model, upside-down bathtubs, hazard rate function, maximum likelihood estimation.*

1.0 Introduction

Oftentimes the addition of an extra shape parameter(s) induced more flexibility to distribution functions mainly for data analysis purposes which improve the modeling potential of the classical distribution. To mention a few, Azzalini (1985) studied the skew-normal distribution by the addition of an extra parameter to the normal distribution to induce more flexibility into the normal distribution. Mudholkar and Srivastava (1993) developed method that introduced an extra shape parameter to the Weibull distribution and called it exponentiated Weibull model which consist two shape parameters and one scale parameter. Marshall and Olkin (1997) developed another method that can be used to increase the parameter(s) of any standard probability distribution. The well-known generators are the following: the beta-G family of distribution which was developed by Eugene et al. (2002), Cordeiro and de Castro (2011) developed the Kumaraswamy-G family of distribution, exponentiated generalized-G family of distribution was studied by Cordeiro et al. (2013). Hassan and Eligarhy (2016) developed the Kumaraswamy Weibull generated family of distributions, the Odd generalized exponential family of distribution was proposed and studied Alizadeh et al. (2017), the exponentiated Weibull-H family of distribution was developed by Cordeiro et al. (2017), exponentiated generalized-G Poisson family of distribution was developed and studied by Aryal and Yousof (2017). Marshall-Olkin generalized-G Poisson family of distribution was developed and studied by Korkmaz et al. (2018). Oluyede, et al. (2018) developed the gamma Weibull-G family of distributions by combining the gamma generator with the Weibull-G family of distributions which was defined by Bourguignon et al. (2014) and odd Lomax-G family of distribution was studied by Cordeiro et al. (2019). Recently, the alpha power transformation was proposed and studied by Mahdavi and Kundu (2017).

Let $G(x)$ represent the cumulative distribution function (cdf) of any continuous random variable X, then CDF of Alpha Power Transformed (APT) family is given by

$$F(x) = \begin{cases} \frac{\alpha^{G(x)} - 1}{\alpha - 1}, & if\ \alpha > 0, \alpha \neq 1 \\ G(x), & \alpha = 0 \end{cases} \tag{1}$$

And the associated probability density function (pdf) is

$$f(x) = \begin{cases} \frac{log\alpha}{\alpha - 1} g(x)\alpha^{G(x)}, & if\ \alpha > 0, \alpha \neq 1 \\ G(x), & if\ \ \alpha = 0 \end{cases} \tag{2}$$

The transformation has been widely used by researchers to obtain alpha transformed distributions. Namely, Dey et al. (2017a, 2017b, 2018, 2019) examined the properties of the new extensions of generalized exponential distribution with an application to ozone data, a new extension of Weibull distribution with application to real-life data, extended Weibull distribution with application to real-life data, alpha transformed inverse Lindley distribution which exhibits upside-down bathtub shape failure rate, and alpha power transformed Lindley distribution with applications to earthquake data. Hassan et al. (2018) investigate the properties of alpha power transformed extended exponential distribution, alpha power Weibull distribution was studied by Nasser et al. (2017). Ogunde et al. (2020a, 2020b) studied the properties of alpha power extended Bur II distribution and alpha power extended inverted Weibull distribution respectively.

We derived our motivation from the advantages offered by a generalized distribution which are relevant in modeling lifetime data that are non-monotonic exhibiting different shapes of the hazard function ranges from increasing, decreasing, and bathtub shapes, as well as the versatility of compounding alpha g family of distribution with exponentiated Inverted Weibull distribution in modeling real-life data. Here, we study a new generalization called the Alpha Power Exponentiated Gumbel type-2 ($APEGT - 2$) distribution which possesses these properties.

We are also motivated to study the $APEGT - 2$ distribution because of its simplicity and extensive usage of Gumbel type-2 ($GT - 2$) distribution in modeling lifetime events. Also, the current generalization promotes a wider application even to complex situations that involve different shapes of the hazard function.

2.0 The model, sub-models, and properties of $APEGT - 2$ model

The probability density function (pdf) and the associated distribution function (cdf) of the two-Exponentiated gumbel distribution was developed and study by Okorie et al. (2016) and are given by

$$g(x; \omega, \rho) = \omega\rho\theta x^{-\omega-1} e^{-\theta x^{-\omega-1}} \left(1 - e^{-\theta x^{-\omega}}\right)^{\rho-1}, \qquad x > 0 \tag{3}$$

and

$$G(x; \omega, \rho) = 1 - \left(1 - e^{-\theta x^{-\omega}}\right)^{\rho}, \qquad x > 0 \tag{4}$$

Where $\omega, and\ \rho$ are positive sha[e parameters and θ is a positive scale parameter. Several generalizations of the Gumbel type-2 distribution have been developed and studied, see, Okorie et al. (2016) proposed and studied the properties of an exponentiated form of the GTT distribution of Lehman type I. Okorie et al. (2017) investigated the properties of the Kumaraswamy G

Exponentiated Gumbel type-two distribution, Ogunde et al. (2020) developed and studied the four-parameter extended Gumbel type-2 distribution among many others.

Given that $G(x)$ is the cdf of a distribution given in (4), then inserting (4) in (1) gives another distribution called the APEGT-2 distribution which cdf is given by

$$F(x;\alpha,\omega,\rho,\theta)=\begin{cases}\dfrac{\alpha^{1-\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}-1}{\alpha-1}, & if\ \alpha>0,\alpha\neq 1\\ \alpha^{\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}, & \alpha=0\end{cases} \tag{5}$$

And the corresponding pdf to (5) is given by

$$f(x)=\begin{cases}\dfrac{log\alpha}{\alpha-1}\omega\rho\theta x^{-\omega-1}e^{-\theta x^{-\omega}}\left(1-e^{-\theta x^{-\omega}}\right)^{\rho-1}\alpha^{1-\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}, & if\ \alpha,>0,\alpha\neq 1\\ \omega\rho\theta x^{-\omega-1}e^{-\theta x^{-\omega-1}}\left(1-e^{-\theta x^{-\omega}}\right)^{\rho-1}, & if\ \ \alpha=0\end{cases} \tag{6}$$

Where ω, ρ and α are positive shape parameters and θ is a positive scale parameter. The plots of the distribution and the density functions is given in Figures 1 and 2.

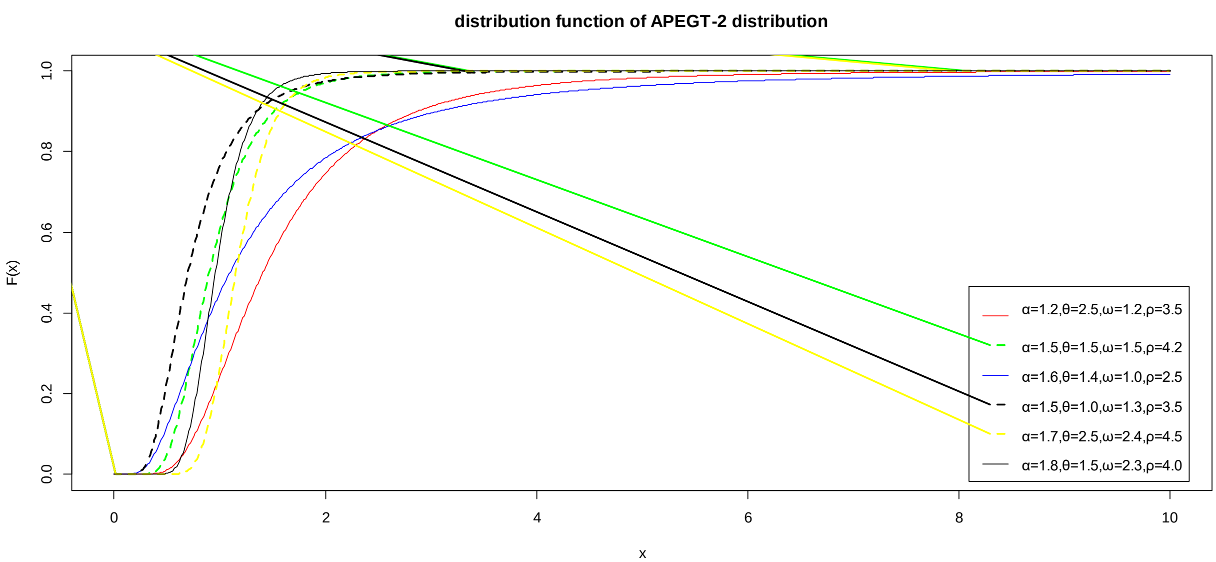

Figure 1. Graph of the distribution function of APEGT-2 distribution

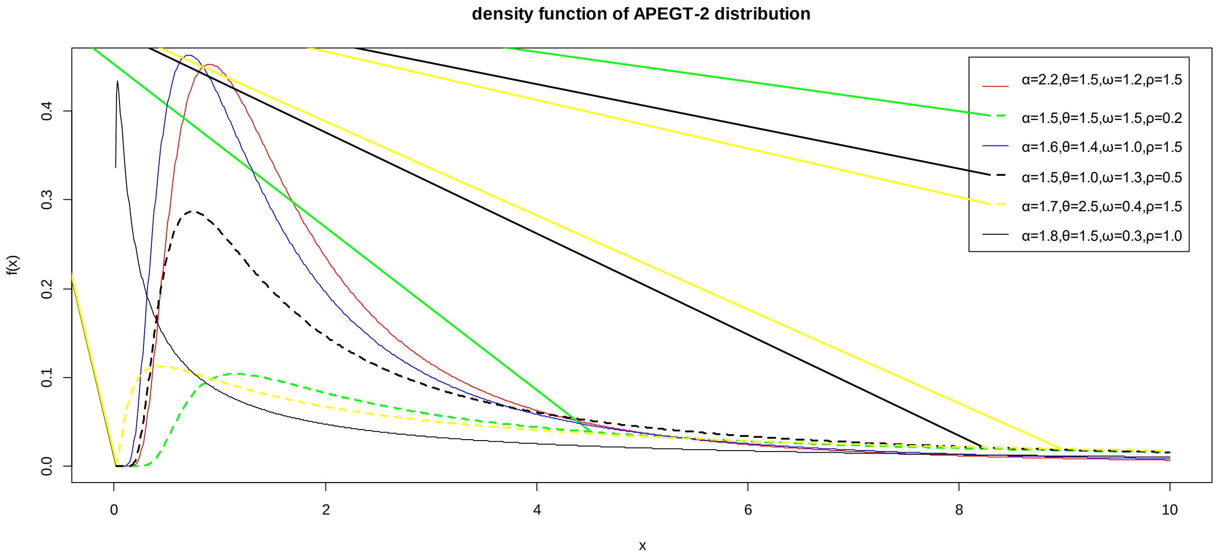

Figure 2. Graph of the density function of APEGT-2 distribution

The survival function ($S(x)$), hazard function ($h(x)$), reversed hazard function ($r(x)$), and the cumulative hazard function ($\zeta(x)$) of the $APEGT-2$ distribution are respectively given by

$$S(x;\alpha,\omega,\rho,\theta)=1-\frac{\alpha^{1-\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}-1}{\alpha-1}\quad ,x>0, \tag{7}$$

$$h(x;\alpha,\omega,\rho,\theta)=\frac{\frac{log\alpha}{\alpha-1}\omega\rho\theta x^{-\omega-1}e^{-x^{-\theta\omega-1}}\left(1-e^{-\theta x^{-\omega}}\right)^{\rho-1}\alpha^{1-\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}}{1-\frac{\alpha^{\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}-1}{\alpha-1}}\ ,\ \ x>0, \tag{8}$$

and

$$r(x;\alpha,\omega,\rho,\theta)=\frac{\frac{log\alpha}{\alpha-1}\omega\rho\theta x^{-\omega-1}e^{-\theta x^{-\omega-1}}\left(1-e^{-\theta x^{-\omega}}\right)^{\rho-1}\alpha^{1-\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}}{\frac{\alpha^{\left(1-e^{-\theta x^{-\omega}}\right)^{\rho}}-1}{\alpha-1}}\ ,\ \ x>0. \tag{9}$$

The graph of the cdf and the pdf of $APEGT-2$ distribution is given in figure 1 and that of the $h(x;\alpha,\omega,\rho)$ in figure 2. In particular, figure 2 demonstrate the flexibility of $APEGT-2$ model in modeling different kinds of data exhibiting different shapes of the hazard function. We observe that the graph of the $h(x;\alpha,\omega,\rho)$ of $APEGT-2$ is decreasing, increasing, upside-down bathtub.

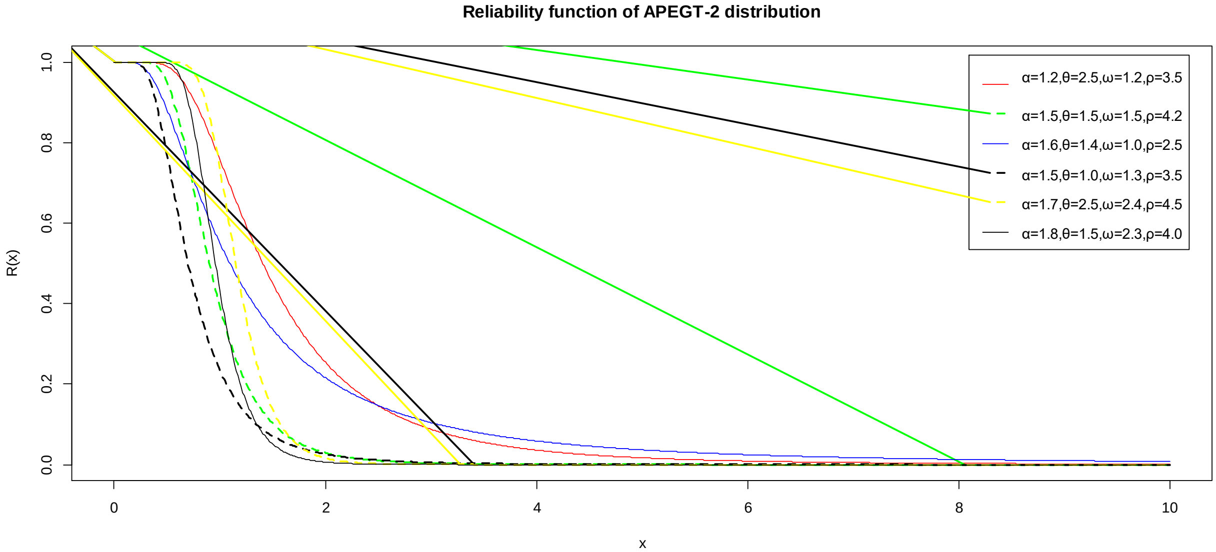

Figure 3. Graph of the reliability function of APEGT-2 distribution

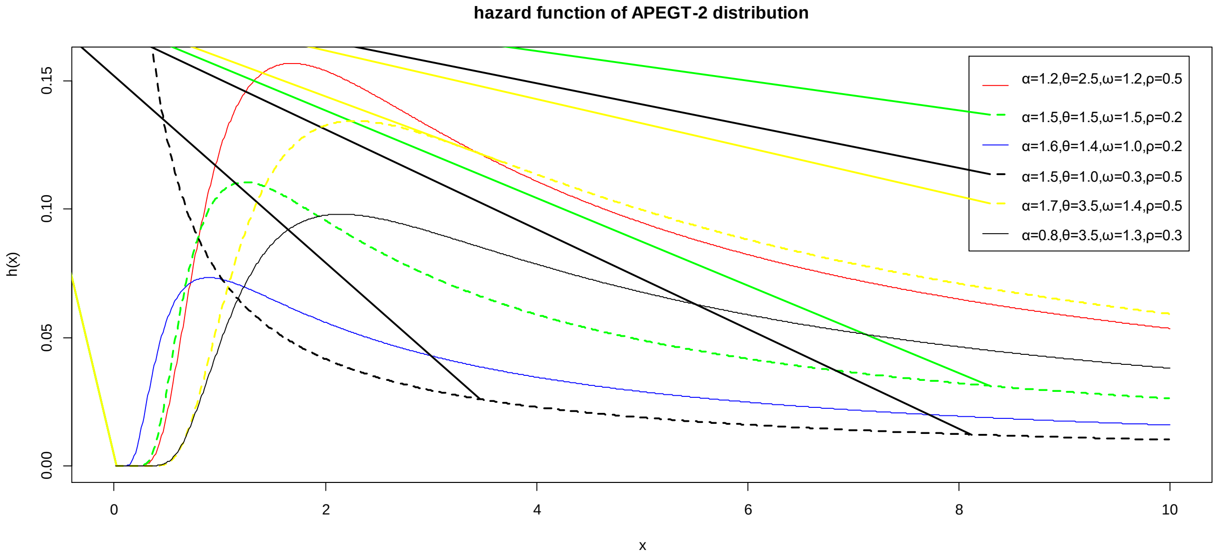

Figure 4. Graph of the hazard function of APEGT-2 distribution

2.1 Quantile function

Quantile function can be defined as an inverse of the distribution function. Consider the relation

$$F(X) = U \Rightarrow X = F^{-1}(U)$$

Where U follows standard Uniform distribution. The q^{th} quantile of $APEGT-2$ distribution is given by

$$X_q = \left\{-\frac{1}{\theta}\left[log\left(\frac{1}{log\alpha}[1 - log(1 + (\alpha - 1)u)]\right)^{1/\rho}\right]\right\}^{-1/\omega} \tag{10}$$

The lower quartile, mean, and the upper quartile $APEGT-2$ distribution can be obtained from (10) by setting the value of q to be 0.25, 0.5, and 0.75 respectively. An expression for the lower quartile, median, and upper quartile is given as

$$X_{0.25} = \left\{-\frac{1}{\theta}\left[log\left(\frac{1}{log\alpha}\left[1-log\big(1+0.25(\alpha-1)\big)\right]\right)^{1/\rho}\right]\right\}^{-1/\omega} \tag{11}$$

$$X_{0.5} = \left\{-\frac{1}{\theta}\left[log\left(\frac{1}{log\alpha}\left[1-log\big(1+0.5(\alpha-1)\big)\right]\right)^{1/\rho}\right]\right\}^{-1/\omega}$$

and

$$X_{0.75} = \left\{-\frac{\mathbf{1}}{\boldsymbol{\theta}}\left[log\left(\frac{1}{log\alpha}\left[1-log\big(1+0.75(\alpha-1)\big)\right]\right)^{1/\rho}\right]\right\}^{-1/\omega} \tag{12}$$

2.2 Random numbers generation

Random numbers can be generated for the $APEGT-2$ $(\alpha,\rho\,,\omega,\theta)$ distribution, for this let, simulating values of random variable X with the cdf given in (5) and u denote a uniform random variable in (0, 1), then the simulated values of X are obtained by as,

$$X = \left\{-\frac{1}{\theta}\left[log\left(\frac{1}{log\alpha}\left[1-log(1+(\alpha-1)u)\right]\right)^{1/\rho}\right]\right\}^{-1/\omega} \tag{13}$$

2.3 Mixture representation for the density function

The mixture representation of the density function is a very useful tool used in deriving the statistical properties of generalized distribution. In this section, the mixture representation of the $APEGT-2$ density function is obtained. Using the following series representation:

$$\alpha^m = \sum_{t=0}^{\infty}\frac{(log\alpha)^t}{t!}m^t \tag{14}$$

$$(1-v)^y = \sum_{t=0}^{\infty}(-1)^t\binom{y}{t}v^t \tag{15}$$

Using the series expansion given in (14) and (15) in (6), we obtain a mixture representation of the pdf of APEGT-2 distribution as

$$f(x) = \frac{\omega\rho\theta}{\alpha-1}\sum_{i=j=k=0}^{\infty}\frac{(log\alpha)^i}{i!}(-1)^{j+k}\binom{i}{j}\binom{\rho(i+1)}{k}x^{-\omega-1}e^{-(k+1)\theta x^{-\omega}} \tag{16}$$

The above expression is a density of the $GT-2$ distribution with scale parameter $(k+1)\theta$ and shape parameter ω.

3.0 Ordinary and incomplete moment

The ordinary moments of distribution play a very important role in statistical applications. The r^{th} moment of a random variable X can be obtained using

$$E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx \tag{17}$$

Putting (16) in (17), we have

$$\mu_r' = \frac{\omega\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} f^l \tag{18}$$

where

$$f^l = \int_{-\infty}^{\infty} x^{r-\omega-1} e^{-(1+k)\theta x^{-\omega}} \, dx \tag{19}$$

By letting $z = (1+k)\theta x^{-\omega}, x = z^{-\frac{1}{\omega}}\left((1+k)\theta\right)^{\frac{1}{\omega}}$ and putting it in (19), we have

$$f^l = \frac{1}{\omega} \theta^{\frac{r}{\omega}} (k+1)^{\frac{r}{\omega}} \Gamma(1 - {}^{r}/_{\omega})$$

Finally r^{th} moment of $APEGT-2$ distribution is given by

$$\mu_r' = \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} \theta^{1+\frac{r}{\omega}} (k+1)^{\frac{r}{\omega}} \Gamma(1 - {}^{r}/_{\omega}) \tag{20}$$

$r < \omega$. For $r = 1,2, \ldots$ $\Gamma(.)$ is the gamma function. By taking $r = 1$, we obtain the mean of X that is, $\mu_1' = \mu$. The variance of X obtained by $\sigma^2 = E[(X-\mu)^2] = \mu_2' - \mu^2$. Also, we can determine the r^{th} central moment and r^{th} cumulant of X respectively defined by

$$\mu_r = E[(X-\mu)^r] = \sum_{h=0}^{r} \binom{r}{h} \mu_{r-h}' (-1)^h \mu^h, \qquad k_r = \mu_r' - \sum_{h=1}^{r-1} \binom{r-1}{h-1} k_h \mu_{r-h}',$$

Taking $k = \mu$, several measures of skewness and kurtosis based on the central moments (or cumulants) can be obtained.

An expression for an Incomplete moment is given by

$$\varphi_r(t) = \int_0^t x^r f(x)dx \tag{21}$$

Putting (16) in (21), we have

$$\boldsymbol{\varphi_r(t)} = \frac{\theta\omega\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (-1)^k f^*$$

where

$$f^* = \int_0^t x^{r-\omega-1} e^{-(1+k)\theta x^{-\omega}} \, dx \tag{22}$$

Also, by letting $z = (1+k)\theta x^{-\omega}, x = z^{-\frac{1}{\omega}}\left((1+k)\theta\right)^{\frac{1}{\omega}}$ and putting it in (22), we have

$$f^l = \frac{1}{\omega} \left((1+k)\right)^{\frac{r}{\omega}} \Gamma(1 - {}^{r}/_{\omega}, (1+k)t^{-\omega})$$

Finally the r^{th} incomplete moment of $APEGT-2$ distribution is given by

$$\varphi_r(t) = \frac{\rho}{\alpha-1}\sum_{i=j=k=0}^{\infty}\frac{(log\alpha)^i}{i!}(-1)^{j+k}\binom{i}{j}\binom{\rho(i+1)}{k}\theta^{1+\frac{r}{\omega}}(k+1)^{\frac{r}{\omega}}\Gamma\{1-{}^{r}/_{\omega},(1+k)t^{-\omega}\}$$
(23)

Where $\Gamma(l,n) = \int_n^{\infty} v^{l-1}e^{-v}\,dv$ is the complementary incomplete gamma function. The first incomplete moment of $APEGT-2$ distribution is given as

$$\varphi_1(t) = \frac{\rho}{\alpha-1}\sum_{i=j=k=0}^{\infty}\frac{(log\alpha)^i}{i!}(-1)^{j+k}\binom{i}{j}\binom{\rho(i+1)}{k}\theta^{1+\frac{r}{\omega}}(k+1)^{\frac{1}{\omega}}\Gamma\{1-{}^{1}/_{\omega},(1+k)t^{-\omega}\}$$
(24)

3.1 Maximum likelihood Estimation: Suppose a random sample of $x_1, x_2, \dots, x_n$ from the $APEGT-2$ distribution, the likelihood function for $z = (\alpha, \omega, \rho, \theta)$ is

$$L(\underline{x},\xi) = \prod_{i=1}^{n}\frac{log\alpha}{\alpha-1}\omega\rho\theta x^{-\omega-1}e^{-x^{-\omega-1}}\left(1-e^{-x^{-\omega}}\right)^{\rho-1}\alpha^{1-\left(1-e^{-x^{-\omega}}\right)^{\rho}} \quad (25)$$

And the log-likelihood function $logL(\underline{x},z) = l$ is presented as

$$l = nlog(\rho) + nlog(\omega) + nlog(\theta) + nlog\left(\frac{log\alpha}{\alpha-1}\right) - (\omega+1)\sum_{i=1}^{n}log\,(x_i) - \sum_{i=1}^{n}x_i^{-\omega-1}$$
$$-(\rho-1)\sum_{i=1}^{n}log\left(1+e^{-\theta x_i{}^{-\omega}}\right) + log\alpha\sum_{i=1}^{n}1-\left(1-e^{-\theta x_i{}^{-\omega}}\right)^{\rho} \quad (26)$$

We differentiate (26) with respect α, ρ and ω, to obtain the element of the score vector $\left(U_\alpha = \frac{\partial l}{\partial\alpha}, U_\lambda = \frac{\partial l}{\partial\rho}, U_\omega = \frac{\partial l}{\partial\omega}, U_\theta = \frac{\partial l}{\partial\theta}\right)^T$. The elements of the score vector are given by

$$U_\rho = \frac{n}{\rho} - \sum_{i=1}^{n}log\left(1+e^{-\theta x_i{}^{-\omega}}\right) + \sum_{i=1}^{n}\left(1-e^{-\theta x_i{}^{-\omega}}\right)^{\rho}log\left(1-e^{-\theta x_i{}^{-\omega}}\right) \quad (27)$$

$$U_\omega = \frac{n}{\omega} - \sum_{i=1}^{n}log\,(x_i) + (\omega+1)\sum_{i=1}^{n}x_i^{-\omega-1} + (\rho-1)\sum_{i=1}^{n}\frac{x_i{}^{-\omega-1}e^{-\theta x_i{}^{-\omega}}}{\left(1+e^{-\theta x_i{}^{-\omega}}\right)} \quad (28)$$

$$U_\alpha = \frac{n}{\alpha-1} + \frac{n(\alpha-1-\alpha\log(\alpha))}{\alpha(\alpha-1)\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-e^{-\theta x_i{}^{-\omega}}\right)^{\rho} \quad (29)$$

4.0 Practical Application of APEGT-2 model

In this section, the $APEGT-2$ distribution is compared with Alpha Power Exponentiated Inverse Exponential (APEIE), Alpha Power Exponentiated inverted Weibull (APEIW), and Gumbel Type-2 ($GT-2$) distributions. Different goodness of fit measures like Kolmogorov- Smirnov (K-S) statistics, Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) are obtained using R-package for real data sets: fracture toughness, taxes revenue's data and coal mining disasters data. The better fit corresponds to smaller, AIC, CAIC, BIC, $k-s$ and $-l$ value. The Maximum Likelihood Estimates (MLEs) of the unknown parameters and values of goodness of fit measures are computed for $APEGT-2$ distribution and its sub-models

The data set is the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The data was provided and studied in Andrews and Herzberg (2012). The data set are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960. Table 1 shows that the epoxy data is positively skewed, leptokurtic, and over-dispersed. The graph of total time on test (TTT) and the violin plots indicates that the epoxy data exhibits a nonmonotone failure rate is also positively skewed.

Table 1 Summary statistics for the Epoxy data set

q_1	$Median$	$mean$	q_3	$range$	$Variance$	$Kurtosis$	$Skewness$
0.905	1.736	1.959	2.296	9.065	2.477	8.161	1.979

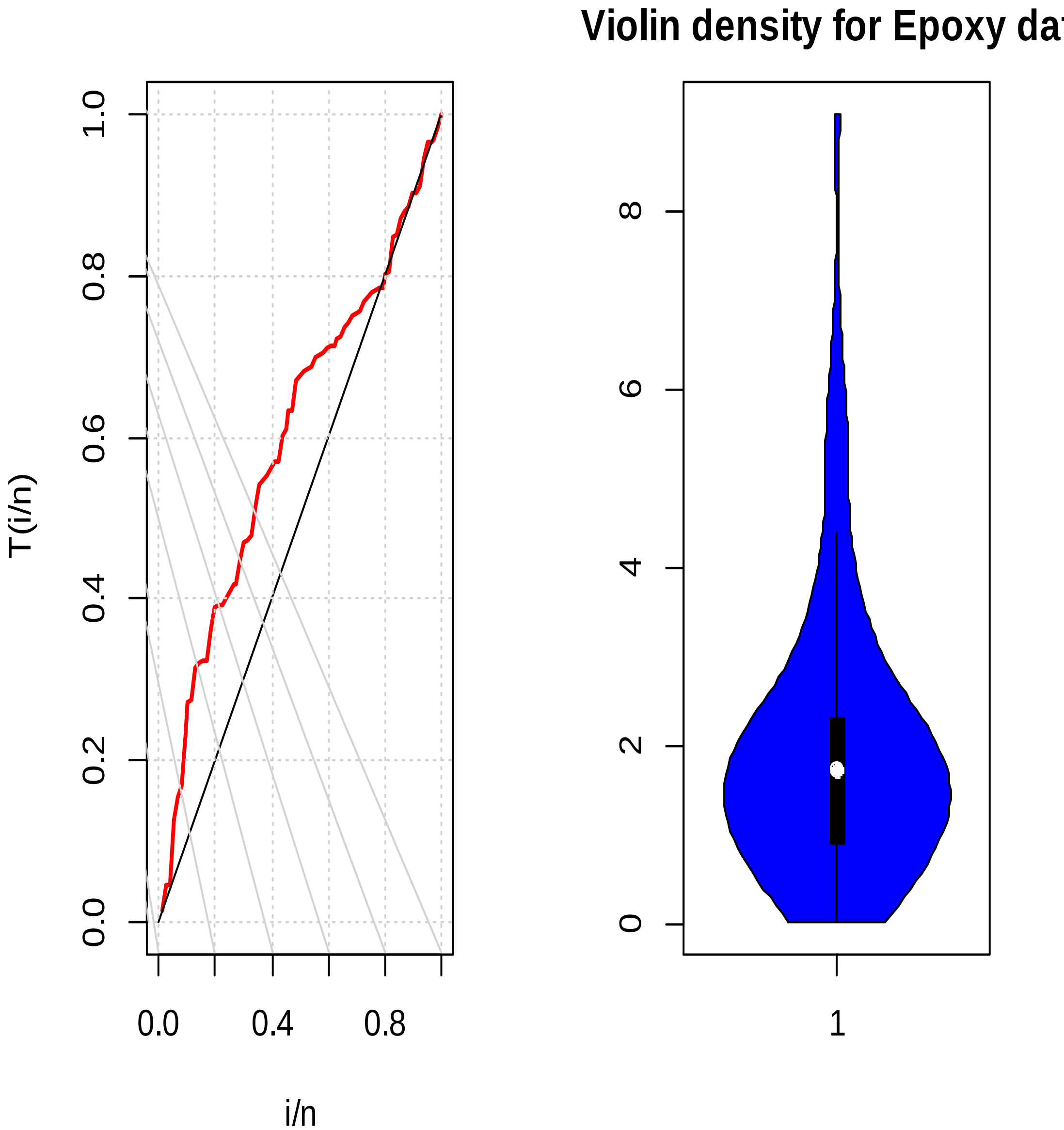

Figure 5.0 plot of TTT plot and Violin plot

Table 2: Analytical results of the $APEGT-2$ model and other competing models for Epoxy data

$Model$	α	ω	ρ	θ	$-l$	AIC	$CAIC$	BIC	k
$APEGT-2$	11.81 (7.32)	18.67 (11.65)	0.34 (0.06)	2.99 (0.67)	124.27	256.54	257.10	265.86	0.101
$APEIW$	20.84 (9.63)	2.75 (0.28)	$-$ $(-)$	0.63 (0.05)	132.79	271.58	271.91	278.57	0.143
$APEIE$	17.51 (7.08)	1.20 (0.16)	$-$ $(-)$	0.39 (0.08)	144.55	295.11	295.44	302.10	0.007
$GT-2$	$-$ $(-)$	0.86 (0.11)	0.76 (0.05)	$-$ $(-)$	153.54	311.08	311.24	315.74	0.190

The new $APEGT-2$ model is much better than other three important competitive models with smallest value of AIC, $CAIC$, BIC, and k value in modeling the second data set.

5.0 Concluding Remarks

We have developed studied the $APEGT-2$ distribution along with its properties such as: descriptive measures based on the quantiles, moments, incomplete moments, Lorenz and Bonferroni curves, stress-strength reliability, weighted moment, entropy, and order statistics. Maximum Likelihood estimates are computed. To illustrate the performance of the MLEs, a lifetime data sets was used. Application of the APEGT-2 model to Epoxy data is presented to demonstrate its tractability and flexibility in modeling real life data and have shown that the APEGT-2 distribution is empirically better for modeling epoxy data than all other models considered in this study.

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